International Journal of Theoretical Physics, Vol. 38, No. 4, 1999

Primordial Black Holes: Pair Creation, Lorentzian Condition, and Evaporation

Raphael Bousso[1] and Stephen W. Hawking[2]

Received September 15, 1998

The wave function of the universe is usually taken to be a functional of the three-metric on a spacelike section, Σ, which is measured. It is sometimes better, however, to work in the conjugate representation, where the wave function depends on a quantity related to the second fundamental form of Σ. This makes it possible to ensure that Σ is part of a Lorentzian universe by requiring that the argument of the wave function be purely imaginary. We demonstrate the advantages of this formalism first in the well-known examples of the nucleation of a de Sitter or a Nariai universe. We then use it to calculate the pair creation rate for submaximal black holes in de Sitter space, which had been thought to vanish semiclassically. We also study the quantum evolution of asymptotically de Sitter black holes. For black holes whose size is comparable to that of the cosmological horizon, this process differs significantly from the evaporation of asymptotically flat black holes. Our model includes the one-loop effective action in the s-wave and large-N approximation. Black holes of the maximal mass are in equilibrium. Unexpectedly, we find that nearly maximal quantum Schwarzschild–de Sitter black holes antievaporate. However, there is a different perturbative mode that leads to evaporation. We show that this mode will always be excited when a pair of maximal cosmological black holes nucleates.

1. INTRODUCTION

The no-boundary proposal [1] is formulated in terms of Euclidean path integrals. But the world we live in is Lorentzian, or at least we interpret our observations in terms of Lorentzian spacetime. One therefore has to continue the results from the Euclidean path integrals analytically to the Lorentzian regime.

[1] Department of Physics, Stanford University, Stanford, California 94305-4060.
[2] Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge, CB3 9EW, U.K.

0020-7748/99/0400-122 7$16.00/0

The approach to quantum cosmology that has been followed in the past is to examine the behavior of the wave function as a function of the overall scale a of the metric h_{ij} on the spacelike surface Σ. If the dependence on a was exponential, this was interpreted as corresponding to a Euclidean spacetime, while an oscillatory dependence on a was interpreted as corresponding to a Lorentzian spacetime.

For example, in the case of Einstein gravity with a cosmological constant Λ, the path integral for the wave function of a three-sphere of radius a will be dominated by an instanton which is part of a four-sphere of radius $R_0 = \sqrt{3/\Lambda}$. In this saddlepoint approximation, the wave function will be given by e^{-I}, where I is the Euclidean action of the saddlepoint geometry; we are neglecting a prefactor. For $a < R_0$, there will be a real Euclidean geometry bounded by the three-sphere Σ of radius a. The wave function Ψ will be 1 for $a = 0$, and will increase rapidly with a, up to $a = R_0$. For $a > R_0$, there are no Euclidean solutions with the given boundary conditions.

There are, however, two complex solutions, each of which can be thought of as half the Euclidean four-sphere, joined to part of the Lorentzian de Sitter solution. The real part of the action of these complex solutions is equal to the action of the Euclidean half-four-sphere, and is the same for all values of a. On the other hand, the imaginary part of the action comes from the Lorentzian de Sitter part of the solution, and depends on a. Thus the wave function for large a oscillates rapidly with constant amplitude.

This shows the association between an oscillatory wave function and a Lorentzian spacetime, but the distinction between exponential and oscillatory is not precise, and does not identify which part of the wave function describes which physical situation. In more complicated situations, the saddlepoint complex solutions will not separate neatly into Euclidean and Lorentzian parts, so it is not clear how to calculate the probability of Lorentzian geometries.

One might apply appropriate operators to the wave function to recover information about whether a given spacelike surface is part of a Lorentzian or a Euclidean spacetime. But the use of operators is cumbersome and requires the evaluation of Ψ for a range of arguments. It would be preferable if the observable geometric properties, such as the Lorentzian character of the universe, were manifest in the argument of the wave function. The square of its amplitude would then yield a probability measure for any given set of such quantities.

We therefore want to put forward an approach which focuses on the defining characteristic of a Lorentzian geometry in the neighborhood of Σ. This is that the induced metric h_{ij} on Σ should be real, but the second fundamental form,

$$K_{ij} = \nabla_i n_j \tag{1.1}$$

defined for Euclidean signature, should be purely imaginary. Here n^j is the unit normal to the surface Σ. The second fundamental form is also called the extrinsic curvature of the surface Σ in the manifold M. It can be regarded as the derivative of the metric h_{ij} on Σ, as Σ is moved in its normal direction in M. Thus, requiring the second fundamental form to be purely imaginary means that h_{ij} has a real derivative with respect to the Lorentzian time coordinate $t = \mathrm{Im}(\tau)$, where τ is Euclidean time. This is the condition for a Lorentzian geometry in a neighborhood of Σ.

The second fundamental form, K_{ij}, is trivially related to π_{ij}, the momentum conjugate to h_{ij}:

$$\pi_{ij} = -h^{1/2}(K_{ij} - h_{ij} K_{kl} h^{kl}) \tag{1.2}$$

where h is the determinant of the metric h_{ij}. Clearly, for real metrics h_{ij}, taking K_{ij} to be purely imaginary is equivalent to taking π_{ij} purely imaginary. It is easy to transform from the usual representation of the wave function $\Psi[h_{ij}]$ to the momentum representation, in which the wave function is a functional of π_{ij}. The two representations are related by a Laplace transform:

$$\Psi[\pi^{ij}] = \int d[h_{ij}]\, \Psi[h_{ij}] \exp\left(- \int_\Sigma d^3x\, \pi^{ij} h_{ij} \right) \tag{1.3}$$

where the integral over the metric components at each point of Σ is taken to be over all h_{ij} with positive determinant h. This Laplace transform can be analytically continued to complex values of π^{ij}. The wave function for a universe that is Lorentzian in a neighborhood of Σ is then obtained by taking π^{ij} to be purely imaginary.

Thus the requirement that we live in a Lorentzian universe can be made manifest in the argument of the wave function. Further support for choosing the momentum representation comes from the fact that we cannot measure the metric globally on a spacelike section, but that the expansion rate of the universe, which is related to the second fundamental form, is easily observable.

The saddlepoint approximation to the wave function will be

$$\Psi[\pi^{ij}] = e^{-I} \tag{1.4}$$

where we neglect a prefactor; here

$$I = -\frac{1}{16\pi} \int d^4x\, g^{1/2}(R - 2\Lambda) \tag{1.5}$$

is the Euclidean action[3] of a complex solution of the field equations with

[3] Note that this action does not contain the usual surface term, which is canceled exactly in the Laplace transform.

the imaginary given values of π^{ij} on Σ. This complex saddlepoint solution will be Lorentzian near Σ by construction. Further away it may be complex or Euclidean, but this does not matter because one is making measurements only on Σ. One therefore has to perform a path integral over the metric everywhere except on Σ. The use of a complex saddlepoint solution does not mean that spacetime is complex. It can just be regarded as a mathematical trick to evaluate the path integral.

2. HOMOGENEOUS ISOTROPIC UNIVERSE WITHOUT BLACK HOLES

We can illustrate the above discussion by a consideration of general relativity without matter fields but with a cosmological constant Λ. Because we are not interested in gravitational waves, we shall restrict ourselves to spherically symmetric solutions. This means that the second fundamental form K_{ij} has two independent components, K_s and K_l. By a gauge choice, we can consider only cases with K_l constant on Σ.

A homogeneous isotropic universe without black holes is the background with respect to which we have to compare the probability of a universe containing a pair of black holes. This is the familiar de Sitter model, with the Euclidean saddlepoint metric

$$ds^2 = V(r)\, d\hat{\tau}^2 + V(r)^{-1}\, dr^2 + r^2\, d\Omega^2 \tag{2.1}$$

where

$$V(r) = 1 - \frac{\Lambda}{3} r^2 \tag{2.2}$$

We can make a choice of coordinates in which the spacelike surfaces Σ will be round three-spheres. Then the metric takes the form

$$ds^2 = d\tau^2 + a(\tau)^2\, d\Omega_3^2 \tag{2.3}$$

where $d\Omega_3^2 = dx^2 + \sin^2 x\, d\Omega_2^2$ is the metric on the unit three-sphere, and

$$a(\tau) = R_0 \sin\left(R_0^{-1}\tau\right) \tag{2.4}$$

The second fundamental form[4] K_i^j contains only one independent component $K = K_l$, since

[4] As we pointed out in the previous section, we should strictly be working with the canonical momentum π_{ij}. The Lorentzian condition that the argument of the wave function be purely imaginary, however, can equally well be implemented for various combinations of π_{ij} and h_{ij}, such as K_{ij} or K_i^j. Here we are choosing the latter quantity for the sake of clarity, since it leads to rather simple equations. It is straightforward to repeat the treatment using components of π_{ij}.

$$K_l = K_s = \frac{\dot{a}}{a} \tag{2.5}$$

An overdot denotes differentiation with respect to Euclidean time τ. For K real (i.e., Euclidean), there will always be a real Euclidean solution. For positive K, this will be less than half the Euclidean four-sphere of radius R_0 and for K negative, it will be more than half. The action will be

$$I_{\mathrm{dS}}(K) = -\frac{3\pi}{4\Lambda}\left[2 - \frac{(3 + 2K^2)K}{(1 + K^2)^{3/2}}\right] \tag{2.6}$$

The saddlepoint approximation to the wave function, neglecting the prefactor A, will be

$$\Psi(K) = \exp\left[-I_{\mathrm{dS}}(K)\right] \tag{2.7}$$

For $K = 0$, the saddlepoint solution will be half the Euclidean four-sphere and the wave function will be

$$\Psi = \exp\left(\frac{3\pi}{2\Lambda}\right) \tag{2.8}$$

Having calculated the wave function for real K, one can now analytically continue to complex values. Up the imaginary K axis, only the imaginary part of the action will change, as can be seen from Eq. (2.6). Thus, the amplitude of the wave function will remain at the value for $K = 0$ given in Eq. (2.8). But the phase of the wave function will vary rapidly with the imaginary part of K. The wave function for positive imaginary K will be be given by just one of the two complex solutions we had before. It is the one that consists of the half Euclidean four-sphere, joined at the time of minimum radius to an expanding de Sitter solution (see Fig. 1).

Thus this approach separates the expanding and contracting phases of the de Sitter universe, which occur when one looks at the wave function in the h_{ij} representation.

3. UNIVERSE WITH MAXIMAL BLACK HOLES

To get a universe containing black holes, one would like to calculate the probability for a Lorentzian geometry on a spacelike surface Σ with n handles. This would represent an expanding universe, with n pairs of black holes, that inflated from spacetime foam. It seems reasonable to suppose that the probability of n handles is roughly the nth power of the probability of a single handle, with appropriate phase space factors. Thus it is sufficient to consider the relative probabilities for zero and one handles. We shall restrict

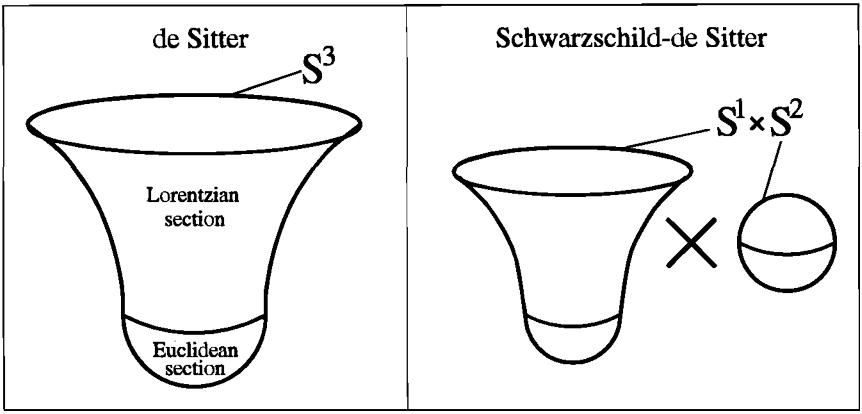

Fig. 1. The creation of a de Sitter universe (left) can be visualized as half of a Euclidean four-sphere joined to a Lorentzian four-hyperboloid. Right: The corresponding nucleation process for a de Sitter universe containing a pair of black holes. In this case the spacelike slices have nontrivial topology.

ourselves to spherical symmetry, to make the problem tractable, but it is reasonable to assume that spherical configurations dominate the path integral.

The zero-handle surfaces (topology S^3) correspond to the Lorentzian de Sitter solution, while the one-handle surfaces (topology $S^1 \times S^2$) correspond to the Schwarzschild–de Sitter solution, with the Lorentzian metric

$$ds^2 = -V(r)\, d\hat{t}^2 + V(r)^{-1}\, dr^2 + r^2\, d\Omega^2 \tag{3.1}$$

where

$$V(r) = 1 - \frac{2\mu}{r} - \frac{\Lambda}{3} r^2 \tag{3.2}$$

This represents a pair of black holes in a de Sitter background. The mass parameter μ of the black holes can be in the range from zero up to a maximum value of $1/(3\sqrt{\Lambda})$. For mass less than the maximum value, the surface gravity of the black hole horizon is greater than that of the cosmological horizon. This means that if one tries to turn the Schwarzschild–de Sitter solution into a compact Euclidean instanton ($d\tau = i\, dt$), one gets a conical singularity either on the black hole horizon or on the cosmological horizon. For this reason, it has been thought that black holes could spontaneously nucleate in a de Sitter background only if they had the maximum mass [2–4]. We shall show in the next section that this conditions can in fact be relaxed.

For now, we shall focus on the maximal case. In this limit, the Schwarzschild–de Sitter solution degenerates into the Nariai solution, in which the two horizons have the same area and surface gravity, and a compact Euclidean instanton is possible without conical singularities:

$$ds^2 = d\tau^2 + a(\tau)^2\, dx^2 + R_1^2\, d\Omega_2^2 \tag{3.3}$$

where $a(\tau) = R_1 \sin(R_1^{-1}\tau)$. The two-spheres on Σ all have the same radius, $R_1 = 1/\sqrt{\Lambda}$, so $K_s = 0$ and there will be only one independent component of the second fundamental form, $K = K_l$. The Euclidean saddlepoint is a direct product of two round two-spheres of radius R_1. The Lorentzian Nariai solution is the direct product of $(1 + 1)$-dimensional de Sitter space with a round two-sphere.

The value of K will govern the size of the first Euclidean two-sphere in the same way it did for the de Sitter four-sphere in the previous section. For real K, the geometry is entirely Euclidean, while for imaginary K, it will consist of half of $S^2 \times S^2$ joined to the expanding half of the Lorentzian Nariai solution (see Fig. 1). The action will be given by

$$I_{\mathrm{N}}(K) = -\frac{\pi}{\Lambda}\left(1 - \frac{K}{\sqrt{1 + K^2}}\right) \tag{3.4}$$

yielding the wave function

$$\Psi_{\mathrm{N}}(K) = \exp[I_{\mathrm{N}}(K)] \tag{3.5}$$

To obtain a Lorentzian universe, we must choose K to be purely imaginary. Then the real part of the Euclidean action, which gives the amplitude of the wave function, will be $-2\pi/\Lambda$. As in the de Sitter case, this is independent of K as long as $\mathrm{Re}(K) = 0$. The imaginary part of the action, which gives the phase of the wave function, depends on K.

To calculate the pair creation rate of Nariai black holes on a de Sitter background, we note that $\Psi^*\Psi$ is a probability measure. It is important to stress that the probability measure depends only on the real part of the saddlepoint action, which stems from the Euclidean sector. In accordance with other instanton methods, the pair creation rate Γ_{N} can thus be obtained by normalizing this probability with respect to de Sitter space:

$$\begin{aligned}\Gamma_{\mathrm{N}} &= \frac{\Psi_{\mathrm{N}}^*\Psi_{\mathrm{N}}}{\Psi_{\mathrm{dS}}^*\,\Psi_{\mathrm{dS}}} = \exp\{-2[\mathrm{Re}(I_{\mathrm{N}}) - \mathrm{Re}(I_{\mathrm{dS}})]\} \\ &= \exp\left(\frac{-\pi}{\Lambda}\right)\end{aligned} \tag{3.6}$$

Therefore the pair creation of black holes is highly suppressed except when the (effective) cosmological constant is close to the Planck value, as it may have been in the earliest stages of inflation.

4. UNIVERSE WITH SUBMAXIMAL BLACK HOLES

In the previous section, we chose to consider only black holes of maximal size in order to avoid a conical singularity in the Euclidean saddlepoint solution. For a metric to dominate the path integral, it has to be a solution of the Einstein equations at every point of the manifold; but on a conical singularity clearly it is not. Thus the action will not be stationary with respect to general variations of a metric containing a conical singularity.

But there is one way to resolve the problem: If the Lorentzian spacelike surface Σ on which the measurements are made can be arranged to contain the conical singularity, then the metric there can be held fixed. Then the four-manifold can be varied only at points where it does solve the Einstein equation, and consequently it will dominate the path integral. This is indeed possible in the Schwarzschild–de Sitter spacetime, as we will show.

Another way to see this is by returning to the analogy that originally motivated the Euclidean prescription for the wave function of the universe. In a quantum mechanical system, the ground-state wave function can be found either as the lowest-eigenvalue solution to the time-independent Schrödinger equation, or as the path integral

$$\Psi_0(x) = N \int \delta x(\tau) e^{-I[x(\tau)]} \tag{4.1}$$

where the integral is over all paths from $x = 0$ at $\tau = -\infty$ to x at $\tau = 0$, and I is their Euclidean action. This can be derived from the Lorentzian propagator

$$\langle x, 0 | 0, t' \rangle = \sum_n \Psi_n(x) \Psi_n(0) e^{iE_n t'} = \int \delta x(t) e^{iS[x(t)]} \tag{4.2}$$

where S is the Lorentzian action and Ψ_n is the eigenfunction to the energy E_n. When one Wick-rotates by taking $\tau = it$, all terms in the sum except for the lowest vanish in the limit as $\tau \to -\infty$, and one obtains Eq. (4.1).

In the semiclassical approximation, Eq. (4.1) becomes

$$\Psi_0(x) = N e^{-I[x_{\rm sol}(\tau)]} \tag{4.3}$$

where $x_{\rm sol}(\tau)$ is a solution of the Euclidean equations of motion with the given boundary conditions (and we assume it is the only one). The corresponding probability measure $\Psi_0^* \Psi_0$ can be obtained by taking the exponential of minus the action of the double of this solution: the path from $x = 0$ at $\tau = -\infty$ to x at $\tau = 0$, and its time reflection from x at $\tau = 0$ to $x = 0$ at $\tau = \infty$. Thus the path will typically have a cusp at $x = 0$. This argument, which is seen to hold in the well-defined case of quantum mechanics, supports our view that cusplike singularities ought to be admitted on the spacelike boundary surface in the Euclidean path integral of quantum cosmology. In this case, too, the probability measure can be obtained both as the square of the amplitude of the wave function or by calculating the action of the double of the saddlepoint path, which is usually nondifferentiable on Σ.

Thus, conical singularities on Σ are allowed, *if they correspond to components of* h_{ij} *that are measured*. For example, if one wants the probability of an $S^1 \times S^2$ handle with a two-sphere cross section σ of area A, one can impose the Lorentzian condition that the real part of the second fundamental form vanish everywhere on Σ except for σ. One cannot specify the second fundamental form on σ, because one is prescribing the metric there. On the other hand, one can impose the Lorentzian condition that the real part of the second fundamental form is zero everywhere else on Σ. This allows one to find a saddlepoint solution bounded by a surface Σ with a handle of area A for any area up to the maximum $4\pi/\Lambda$. Therefore the nucleation of Schwarzschild–de Sitter black hole pairs of any size can be analyzed in the instanton formalism. We choose the cosmological horizon to be regular in the Euclidean sector, which will lead to a conical singularity on the black hole horizon. This is allowed as long as the surface of measurement Σ contains the conical singularity.

The cross section σ corresponds to the black hole horizon; it will be the smallest S^2 in the spacelike surface Σ. (For, assume it is not. Then the σ will not correspond to the conical singularity, whose metric will then not be fixed on the boundary. But such configurations will not dominate in the path integral and can be neglected.) One can now choose some slicing of Schwarzschild–de Sitter spacetime which must have the property that the proper time between points on different slices goes to zero at least quadratically as a function of proper distance from the black hole horizon. This type of slicing is shown schematically in a Penrose diagram in Fig. 2.

It ensures that all Lorentzian spacelike slices will be regular on the black hole horizon. We shall not give any such slicing explicitly. Once a particular slicing is chosen, there will again be only one degree of freedom in the second fundamental form, say $K = \int d^3x\, h^{1/2} K_{ij} h^{ij}$.

Thus, in the Schwarzschild–de Sitter case, the wave function has two arguments, A and K. The first determines the size of the black hole, while the second selects a spacelike slice in the saddlepoint metric. The de Sitter and Nariai cases are included for $A = 0$ and $A = 4\pi/\Lambda$, respectively.

The Euclidean part of the saddlepoint metric has a boundary with zero second fundamental form everywhere except on σ, where it is a delta function. This boundary will split the full Euclidean solution in half in the same way as in the de Sitter and Nariai solutions. This half of the Euclidean geometry will give the real part of the action. Choosing K to be purely imaginary leads to a Lorentzian universe, which once again can be obtained by analytically continuing the Euclidean solution. As for the de Sitter and Nariai solutions,

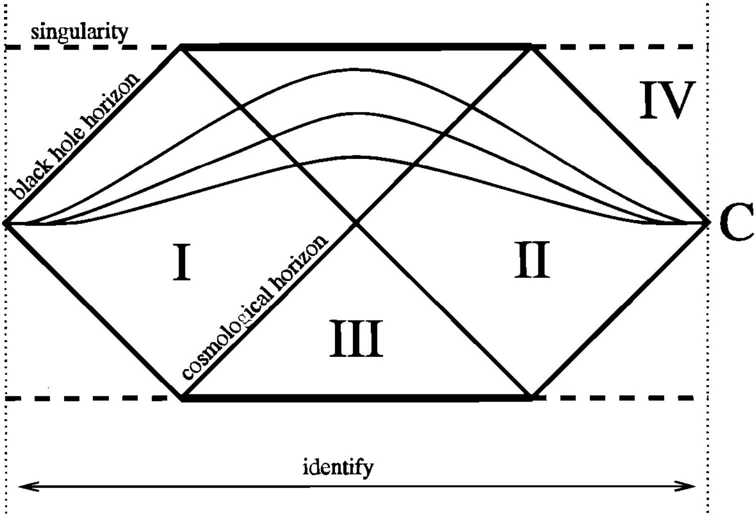

Fig. 2. Penrose diagram of the Schwarzschild–de Sitter spacetime. The point C is the location of the conical singularity in the Euclidean sector. The curved lines indicate a family of spacelike slices which all pass through the conical singularity. This is necessary since one must specify the metric there in order to ensure that the Euclidean solution is a saddlepoint. Regions I and II lie between the black hole and the cosmological horizon. Region III corresponds to an asymptotic de Sitter region, and region IV to the black hole interior.

the Lorentzian section will contribute only to the imaginary part of the action. Therefore the real part of the action will be independent of K for imaginary K:

$$I_{\mathrm{SdS}}\,(A,\,K) = I^{\mathrm{Re}}_{\mathrm{SdS}}\,(A) + iI^{\mathrm{Im}}_{\mathrm{SdS}}\,(A,\,K) \tag{4.4}$$

To calculate the probability measure, and thus the nucleation rate for a Schwarzschild–de Sitter black hole pair, we need only calculate the real part of the action, since

$$\Psi^{*}_{\mathrm{SdS}}\,\Psi_{\mathrm{SdS}} = \exp[-2\,\mathrm{Re}(I_{\mathrm{SdS}})] \tag{4.5}$$

But $2\,\mathrm{Re}(I_{\mathrm{SdS}}) = 2I^{\mathrm{Re}}_{\mathrm{SdS}}(A)$, which is twice the action of the Schwarzschild–de Sitter instanton, which in turn is equal to the action of the full Euclidean Schwarzschild–de Sitter solution, $I^{\mathrm{full}}_{\mathrm{SdS}}$.

Using Eq. (1.5) and $R = 4\Lambda$, one can show that

$$I^{\mathrm{full}}_{\mathrm{SdS}} = -\frac{\Lambda\mathcal{V}}{\forall\pi} - \frac{A\delta}{8\pi} \tag{4.6}$$

where $\mathcal{V}$ is the four-volume of the Euclidean solution. The extra term gives the contribution from a conical deficit angle δ at a two-surface of area A [2].

In order to facilitate the calculation of this action, it is useful to parametrize the Schwarzschild–de Sitter solutions by the radii b and c of the black hole and the cosmological horizon. The parameters Λ and μ can be expressed in terms of the new parameters b and c:

$$\Lambda = \frac{3}{b^2 + c^2 + bc} \tag{4.7}$$

$$\mu = \frac{bc(b + c)}{2(b^2 + c^2 + bc)} \tag{4.8}$$

The Euclidean Schwarzschild–de Sitter metric is

$$ds^2 = V(r)d\tau^2 + V(r)^{-1}\, dr^2 + r^2\, d\Omega^2 \tag{4.9}$$

where $V(r)$ is given by Eq. (3.2); in terms of b and c it takes the form

$$V(r) = \frac{(r - b)(c - r)(r + b + c)}{r(b^2 + c^2 + bc)} \tag{4.10}$$

To avoid a conical singularity at the cosmological (black hole) horizon, the Euclidean time τ must be identified with the period τ_c^{id} (τ_b^{id}), where

$$\tau_{c,b}^{\mathrm{id}} = 2\pi\, \sqrt{g_{rr}}|_{r=c,b} \left| \frac{\partial}{\partial r}\, \sqrt{g_{\tau\tau}} \right|_{r=c,b}^{-1} \tag{4.11}$$

where $g_{\tau\tau} = 1/g_{rr} = V(r)$. This gives

$$\tau_{c,b}^{\mathrm{id}} = 4\pi \left| \frac{\partial V}{\partial r} \right|_{r=c,b} \tag{4.12}$$

We choose to get rid of the conical singularity at $r = c$, so the volume will be

$$\mathcal{V} = \frac{4\pi}{3}(c^3 - b^3)\tau_c^{\mathrm{id}} \tag{4.13}$$

The conical deficit angle at the black hole horizon is by definition

$$\delta = 2\pi\left(1 - \frac{\tau_c}{\tau_b}\right) \tag{4.14}$$

The two-sphere area A is obviously $4\pi b^2$.

With Λ, $\mathcal{V}$, A, and δ expressed in terms of b and c, Eq. (4.6) evaluates to

$$I_{\mathrm{SdS}}^{\mathrm{full}} = -\pi(b^2 + c^2) \tag{4.15}$$

Note that this action is related to the geometric entropy S and the total horizon area in the usual way [5–9]:

$$-I = S = \frac{A + A_c}{4} \tag{4.16}$$

where $A_c = 4\pi c^2$ is the area of the cosmological horizon. Thus we obtain for the pair creation rate of arbitrary-size Schwarzschild–de Sitter black holes in de Sitter space:

$$\Gamma_{\text{SdS}} = \exp[-(I_{\text{SdS}}^{\text{full}} - I_{\text{dS}}^{\text{full}})] = \exp(-\pi bc) \tag{4.17}$$

Using Eqs. (4.7) and $A = 4\pi b^2$, we can easily rewrite this result in terms of Λ and A, the argument we specified in the wave function. However, the physical implications are quite clear from Eq. (4.17): a decreasing cosmological constant corresponds to increasing cosmological horizon size c and thus, as in the maximal case, to increasing suppression. At fixed value of the cosmological constant, the suppression increases with the black hole radius b, which is physically sensible. Considering the Planck length to be the lower bound on the black hole size ($b \geq 1$), we find that even the smallest black holes are highly suppressed unless the cosmological constant is also near the Planck value.

Chao has recently proposed [10] that one should calculate the saddlepoint approximation to the wave function using "constrained instantons," which include spacetimes with a conical singularity. He conjectures the conical singularities should be allowed on the "equator," i.e., the $K_{ij} = 0$ surface on which the real Euclidean geometry is matched to a real Lorentzian one. While our results do not differ from some of those obtained by Chao, we feel that his prescription is rather ad hoc, since it fails to justify why such configurations should dominate the path integral. Moreover, it is ill defined, since the existence of a surface of vanishing second fundamental form is rather special to the simple case of a fixed cosmological constant. In a generic model, the geometry will not be perfectly real anywhere on the tunneling geometry except on the final spacelike surface that is measured [3].

5. QUANTUM EVOLUTION OF SCHWARZSCHILD–DE SITTER BLACK HOLES

Of the effects expected of a quantum theory of gravity, black hole radiance [11] plays a particularly significant role. So far, however, mostly asymptotically flat black holes have been considered. In this work, we investigate a qualitatively different black hole spacetime, in which the black hole is in a radiative equilibrium with a cosmological horizon.

The evaporation of black holes has been studied using two-dimensional toy models, in which one-loop quantum effects were included [12–14]. We have recently shown how to implement quantum effects in a more realistic class of two-dimensional models, which includes the important case of dimensionally reduced general relativity [15]. The result we obtained for the trace anomaly of a dilaton-coupled scalar field will be used here to study the evaporation of cosmological black holes.

We shall consider the Schwarzschild–de Sitter family of black holes. The size of these black holes varies between zero and the size of the cosmological horizon. If the black hole is much smaller than the cosmological horizon, the effect of the radiation coming from the cosmological horizon is negligible, and one would expect the evaporation to be similar to that of Schwarzschild black holes. Therefore we shall not be interested in this case. Instead, we wish to investigate the quantum evolution of nearly degenerate Schwarzschild–de Sitter black holes. The degenerate solution, in which the black hole has the maximum size, is called the Nariai solution [16]. In this solution the two horizons have the same size and the same temperature. Therefore they will be in thermal equilibrium. Intuitively, one would expect any slight perturbation of the geometry to cause the black hole to become hotter than the background. Thus, one may suspect the thermal equilibrium of the Nariai solution to be unstable. The initial stages of such a runaway would be an interesting and novel quantum gravitational effect quite different from the evaporation of an asymptotically flat black hole. In this paper we will investigate whether and how an instability develops in a two-dimensional model derived from four-dimensional general relativity. We include quantum effects at the one-loop level.

The remainder of this article is structured as follows: In Section 6 we review the Schwarzschild–de Sitter solutions and the Nariai limit. We discuss the qualitative expectations for the evaporation of degenerate black holes, which motivate our one-loop study. The two-dimensional model corresponding to this physical situation is presented in Section 7 and the equations of motion are derived. In Section 8 the stability of the quantum Nariai solution under different types of perturbations is investigated. We find, quite unexpectedly, that the Schwarzschild–de Sitter solution is stable, but we also identify an unstable mode. Finally, the no-boundary condition is applied in Section 9 to study the stability of spontaneously nucleated cosmological black holes.

6. EVOLUTION OF NEARLY MAXIMAL BLACK HOLES

6.1. Metric

Recall the Schwarzschild–de Sitter metric

$$ds^2 = -V(r)\,dt^2 + V(r)^{-1}\,dr^2 + r^2\,d\Omega^2 \tag{6.1}$$

where

$$V(r) = 1 - \frac{2\mu}{r} - \frac{\Lambda}{3} r^2 \tag{6.2}$$

For $0 < \mu < \frac{1}{3}\Lambda^{-1/2}$, V has two positive roots r_c and r_b, corresponding to the cosmological and the black hole horizons, respectively. The limit where $\mu \to 0$ corresponds to the de Sitter solution. In the limit $\mu \to \frac{1}{3}\Lambda^{-1/2}$ the size of the black hole horizon approaches the size of the cosmological horizon, and the above coordinates become inappropriate, since $V(r) \to 0$ between the two horizons. Following Ginsparg and Perry [2], we write

$$9\mu^2\Lambda = 1 - 3\epsilon^2, \qquad 0 \leq \epsilon \ll 1 \tag{6.3}$$

Then the degenerate case corresponds to $\epsilon \to 0$. We define new time and radial coordinates ψ and χ by

$$\tau = \frac{1}{\epsilon\sqrt{\Lambda}}\psi; \qquad r = \frac{1}{\sqrt{\Lambda}}\left[1 - \epsilon\cos\chi - \frac{1}{6}\epsilon^2\right] \tag{6.4}$$

In these coordinates the black hole horizon corresponds to $\chi = 0$ and the cosmological horizon to $\chi = \pi$. The new metric obtained from the transformations is, to first order in ϵ,

$$\begin{aligned} ds^2 = &-\frac{1}{\Lambda}\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi\, d\psi^2 \\ &+ \frac{1}{\Lambda}\left(1 - \frac{2}{3}\epsilon\cos\chi\right) d\chi^2 \\ &+ \frac{1}{\Lambda}(1 - 2\epsilon\cos\chi)\, d\Omega_2^2 \end{aligned} \tag{6.5}$$

This metric describes Schwarzschild–de Sitter solutions of nearly maximal black hole size.

In these coordinates the topology of the spacelike sections of Schwarzschild–de Sitter becomes manifest: $S^1 \times S^2$. In general, the radius r of the two-spheres varies along the S^1 coordinate χ, with the minimal (maximal) two-sphere corresponding to the black hole (cosmological) horizon. In the degenerate case, the two-spheres all have the same radius.

6.2. Thermodynamics

The surface gravities of the two horizons are given by [4]

$$\kappa_{c,b} = \sqrt{\Lambda}\left(1 \mp \frac{2}{3}\epsilon\right) + O(\epsilon^2) \tag{6.6}$$

where the upper (lower) sign is for the cosmological (black hole) horizon. In the degenerate case, the two horizons have the same surface gravity and, since $T = \kappa/2\pi$, the same temperature. They are in thermal equilibrium; one could say that the black hole loses as much energy due to evaporation as it gains due to the incoming radiation from the cosmological horizon. Away from the thermal equilibrium, for nearly degenerate Schwarzschild–de Sitter black holes, one could make the simplifying assumption that the horizons still radiate thermally, with temperatures proportional to their surface gravities. This would lead one to expect an instability: By Eq. (6.6), the black hole will be hotter than the cosmological horizon, and will therefore suffer a net loss of radiation energy. To investigate this suspected instability, a two-dimensional model is constructed below, in which one-loop terms are included.

7. TWO-DIMENSIONAL MODEL

The four-dimensional Lorentzian Einstein–Hilbert action with a cosmological constant is

$$S = \frac{1}{16\pi} \int d^4x \, (-g^{\mathrm{IV}})^{1/2} \left[R^{\mathrm{IV}} - 2\Lambda - \frac{1}{2} \sum_{i=1}^{N} (\nabla^{\mathrm{IV}} f_i)^2 \right] \tag{7.1}$$

where R^{IV} and g^{IV} are the four-dimensional Ricci scalar and metric determinant, and the f_i are scalar fields which will carry the quantum radiation.

We shall consider only spherically symmetric fields and quantum fluctuations. Thus, we make a spherically symmetric metric ansatz,

$$ds^2 = e^{2\rho}(-dt^2 + dx^2) + e^{-2\phi} d\Omega^2 \tag{7.2}$$

where the remaining two-dimensional metric has been written in conformal gauge; x is the coordinate on the one-sphere and has the period 2π. Now the spherical coordinates can be integrated out, and the action is reduced to

$$S = \frac{1}{16\pi} \int d^2x \, (-g)^{1/2} \, e^{-2\phi} \left[R + 2(\nabla\phi)^2 \right.$$
$$\left. + 2e^{2\phi} - 2\Lambda - \sum_{i=1}^{N} (\nabla f_i)^2 \right] \tag{7.3}$$

where the gravitational coupling has been rescaled into the standard form. Note that the scalar fields have acquired an exponential coupling to the dilaton in the dimensional reduction. In order to take quantum effects into account, we will find the classical solutions to the action $S + W^*$. Here W^* is the scale-dependent part of the one-loop effective action for dilaton coupled scalars, which we derived in a recent paper [15]:

$$W^* = -\frac{1}{48\pi} \int d^2x\, (-g)^{1/2} \left[\frac{1}{2} R \frac{1}{\Box} R - 6(\nabla\phi)^2 \frac{1}{\Box} R - 2\phi R \right] \tag{7.4}$$

The $(\nabla\phi)^2$ term will be neglected; we justify this neglect at an appropriate place below.

Following Hayward [17], we render this action local by introducing an independent scalar field Z which mimics the trace anomaly. The f fields have the classical solution $f_i = 0$ and can be integrated out. Thus we obtain the action

$$S = \frac{1}{16\pi} \int d^2x\, (-g)^{1/2} \left[\left(e^{-2\phi} + \frac{\kappa}{2} (Z + w\phi) \right) R - \frac{\kappa}{2} (\nabla Z)^2 + 2 + 2e^{-2\phi} (\nabla\phi)^2 - 2e^{-2\phi}\Lambda \right] \tag{7.5}$$

where

$$\kappa \equiv \frac{2N}{3} \tag{7.6}$$

There is some debate about the coefficient of the ϕR term in the effective action. Our result [15] corresponds to the choice $w = 2$; the RST coefficient [13] corresponds to $w = 1$, and the result of Nojiri and Odintsov [18] can be represented by choosing $w = -6$. In ref. 17, probably erroneously, $w = -1$ was chosen. We take the large-N limit, in which the quantum fluctuations of the metric are dominated by the quantum fluctuations of the N scalars; thus, $\kappa \gg 1$. In addition, for quantum corrections to be small we assume that $b \equiv \kappa\Lambda \ll 1$. To first order in b, we shall find that the behavior of the system is independent of w.

For compactness of notation, we denote differentiation with respect to t (x) by an overdot (a prime). Further, we define for any functions f and g

$$\partial f\, \partial g \equiv -\dot{f}\dot{g} + f'g', \qquad \partial^2 g \equiv -\ddot{g} + g'' \tag{7.7}$$

and

$$\delta f\, \delta g \equiv \dot{f}\dot{g} + f'g', \qquad \delta^2 g \equiv \ddot{g} + g'' \tag{7.8}$$

Variation with respect to ρ, ϕ, and Z leads to the following equations of motion:

$$-\left(1-\frac{w\kappa}{4}e^{2\phi}\right)\partial^2\phi + 2(\partial\phi)^2 + \frac{\kappa}{4}e^{2\phi}\partial^2 Z + e^{2\rho+2\phi}(\Lambda e^{-2\phi}-1) = 0 \tag{7.9}$$

$$\left(1-\frac{w\kappa}{4}e^{2\phi}\right)\partial^2\rho - \partial^2\phi + (\partial\phi)^2 + \Lambda e^{2\rho} = 0 \tag{7.10}$$

$$\partial^2 Z - 2\partial^2\rho = 0 \tag{7.11}$$

There are two equations of constraint:

$$\left(1-\frac{w\kappa}{4}e^{2\phi}\right)(\delta^2\phi - 2\delta\phi\,\delta\rho) - (\delta\phi)^2$$
$$= \frac{\kappa}{8}e^{2\phi}\left[(\delta Z)^2 + 2\delta^2 Z - 4\delta Z\,\delta\rho\right] \tag{7.12}$$

$$\left(1-\frac{w\kappa}{4}e^{2\phi}\right)(\phi' - \dot\rho\phi' - \rho'\phi) - \phi\phi'$$
$$= \frac{\kappa}{8}e^{2\phi}\left[\dot Z Z' + 2\dot Z' - 2(\dot\rho Z' + \rho'\dot Z)\right] \tag{7.13}$$

From Eq. (7.11), it follows that

$$Z = 2\rho + \eta \tag{7.14}$$

where η satisfies

$$\partial^2\eta = 0 \tag{7.15}$$

The remaining freedom in η can be used to satisfy the constraint equations for any choice of ρ, $\dot\rho$, ϕ, and ϕ on an initial spacelike section. This can be seen most easily by decomposing the fields and the constraint equations into Fourier modes on the initial S^1. By solving for the second term on the right-hand side of Eq. (7.12), and by using Eqs. (7.14) and (7.15), the first constraint yields one algebraic equation for each Fourier coefficient of η. Similarly, the second constraint yields one algebraic equation for the time derivative of each Fourier coefficient of η. If the initial slice was noncompact, this argument would suffice. Here it must be verified, however, that η and $\dot\eta$ will have a period of 2π. The problem reduces to the question of whether the two constant mode constraint equations can be satisfied. Indeed, while for each oscillatory mode of η there are two degrees of freedom (the Fourier coefficient and its time derivative), the second time derivative of the constant mode coefficient $\ddot\eta_0$ must vanish by Eq. (7.15). Thus there is only one degree of freedom, $\dot{\eta}_0$, for the two constant mode equations. However, since we have introduced no odd modes (i.e., modes of the form sin kx) in the perturbation of ϕ, none of the fields will contain any odd modes. Since each term in Eq. (7.13) contains exactly one spatial derivative, each term will be odd. Therefore all even-mode components of the second constraint vanish identically. In particular the constant mode component will thus be automatically satisfied. Then the freedom in $\dot{\eta}_0$ can be used to satisfy the constant mode component of the remaining constraint, Eq. (7.12), through the first[5] term on the right hand side.

8. PERTURBATIVE STABILITY

8.1. Perturbation Ansatz

With the model developed above we can describe the quantum behavior of a cosmological black hole of the maximal mass under perturbations. The Nariai solution is still characterized by the constancy of the two-sphere radius, $e^{-\phi}$. Because of quantum corrections, this radius will no longer be exactly $\Lambda^{-1/2}$. Instead, the solution is given by

$$e^{2\rho} = \frac{1}{\Lambda_1} \frac{1}{\cos^2 t}, \qquad e^{2\phi} = \Lambda_2 \tag{8.1}$$

where

$$\frac{1}{\Lambda_1} = \frac{1}{8\Lambda} [4 - (w + 2)b + \sqrt{16 - 8(w - 2)b + (w + 2)^2 b^2}] \tag{8.2}$$

$$\Lambda_2 = \frac{1}{2w\kappa} [4 + (w + 2)b - \sqrt{16 - 8(w - 2)b + (w + 2)^2 b^2}] \tag{8.3}$$

Expanding to first order in b, one obtains

$$\frac{1}{\Lambda_1} \approx \frac{1}{\Lambda} \left(1 - \frac{wb}{4}\right) \tag{8.4}$$

$$\Lambda_2 \approx \Lambda \left(1 - \frac{b}{2}\right) \tag{8.5}$$

[5] Note that $\dot{\eta}_0$ can thus be purely imaginary, as indeed it will be for the Nariai solution, signaling negative energy density of the quantum field.

Let us now perturb this solution so that the two-sphere radius $e^{-\phi}$, varies slightly along the one-sphere coordinate x:

$$e^{2\phi} = \Lambda_2[1 + 2\epsilon\sigma(t) \cos x] \tag{8.6}$$

where we take $\epsilon \ll 1$. We will call σ the *metric perturbation*. A similar perturbation could be introduced for $e^{2\rho}$, but it does not enter the equation of motion for σ at first order in ϵ. This equation is obtained by eliminating $\partial^2 Z$ and $\partial^2\rho$ from Eq. (7.9) using Eqs. (7.11) and (7.10), and inserting the above perturbation ansatz. This yields

$$\frac{\ddot{\sigma}}{\sigma} = \frac{a}{\cos^2 t} - 1 \tag{8.7}$$

where

$$a \equiv \frac{2\sqrt{16 - 8(w - 2)b + (w + 2)^2 b^2}}{4 - wb} \tag{8.8}$$

To first order in b, one finds that

$$a \approx 2 + b \tag{8.9}$$

which means that w and therefore the ϕR term in the effective action play no role in the horizon dynamics at this level of approximation. This is also the right place to discuss why the term $\sqrt{-g}(\nabla\phi)^2(1/\Box)R$ in the effective action can be neglected. In conformal coordinates this term is proportional to $(\partial\phi)^2\rho$. Thus, in the ρ-equation of motion, Eq. (7.9), it will lead to a $(\partial\phi)^2$ term, which is of second order in ϵ and can be neglected. In the ϕ-equation of motion, Eq. (7.10), it yields terms proportional to κ that are of first order in ϵ. They will enter the equation of motion for σ via the $\kappa e^{2\phi}\, \partial^2 Z$ term in Eq. (7.10). Thus they will be of second order in b and can be dropped. The neglect of the log μ^2 term [15] can be justified in the same way.

8.2. Horizon Tracing

In order to describe the evolution of the black hole, one must know where the horizon is located. The condition for a horizon is $(\nabla\phi)^2 = 0$. Equation (8.6) yields

$$\frac{\partial\phi}{\partial t} = \epsilon\dot{\sigma} \cos x, \qquad \frac{\partial\phi}{\partial x} = -\epsilon\sigma \sin x \tag{8.10}$$

Therefore, the black hole and cosmological horizons are located at

$$x_b(t) = \arctan\left|\frac{\dot{\sigma}}{\sigma}\right|, \qquad x_c(t) = \pi - x_b(t) \tag{8.11}$$

To first order in ϵ, the size of the black hole horizon r_b is given by

$$r_b(t)^{-2} = e^{2\phi[t, x_b(t)]} = \Lambda_2[1 + 2\epsilon\delta(t)] \tag{8.12}$$

where we define the *horizon perturbation*

$$\delta \equiv \sigma \cos x_b = \sigma\left(1 + \frac{\dot{\sigma}^2}{\sigma^2}\right)^{-1/2} \tag{8.13}$$

We will focus on the early time evolution of the black hole horizon; the treatment of the cosmological horizon is completely equivalent.

To obtain explicitly the evolution of the black hole horizon radius, $r_b(t)$, one must solve Eq. (8.7) for $\sigma(t)$, and use the result in Eq. (8.13) to evaluate Eq. (8.12). If the horizon perturbation grows, the black hole is shrinking: this corresponds to evaporation. It will be shown below, however, that the behavior of $\delta(t)$ depends on the initial conditions chosen for the metric perturbation, σ_0 and $\dot{\sigma}_0$.

8.3. Classical Evolution

As a first check, one can examine the classical case, $\kappa = 0$. This has $a = 2$, and Eq. (8.7) can be solved exactly. From the constraint equations (7.12) and (7.13) it follows that

$$\dot{\sigma} = \sigma \tan t \tag{8.14}$$

Therefore the appropriate boundary condition at $t = 0$ is $\dot{\sigma}_0 = 0$. The solution is

$$\sigma(t) = \frac{\sigma_0}{\cos t} \tag{8.15}$$

Then Eq. (8.13) yields

$$\delta(t) = \sigma_0 = \text{const} \tag{8.16}$$

Since the quantum fields are switched off, no evaporation takes place; the horizon size remains that of the initial perturbation. This simply describes the case of a static Schwarzschild–de Sitter solution of nearly maximal mass, as given in Eq. (6.6).

8.4. Quantum Evolution

When we turn on the quantum radiation ($\kappa > 0$) the constraints no longer fix the initial conditions on the metric perturbation. There will thus be two linearly independent types of initial perturbation. The first is the one we were forced to choose in the classical case: $\sigma_0 > 0$, $\dot{\sigma}_0 = 0$. It describes the spatial section of a quantum-corrected Schwarzschild–de Sitter solution of nearly maximal mass. Thus one might expect the black hole to evaporate. For $a > 2$, Eq. (8.7) cannot be solved analytically. Since we are interested in the early stages of the evaporation process, however, it will suffice to solve for σ as a power series in t. Using Eq. (8.13), one finds that

$$\begin{aligned}\delta(t) &= \sigma_0 \left[1 - \frac{1}{2}(a-1)(a-2)t^2 + O(t^4) \right] \\ &\approx \sigma_0 \left[1 - \frac{1}{2} bt^2 \right] \end{aligned} \tag{8.17}$$

The horizon perturbation shrinks from its initial value. Thus, the black hole size *increases*, and the black hole grows, at least initially, back toward the maximal radius. One could say that nearly maximal Schwarzschild–de Sitter black holes "antievaporate."

This is a surprising result, since intuitive thermodynamic arguments would have led to the opposite conclusion. The antievaporation can be understood in the following way. By specifying the metric perturbation, the radiation distribution of the Z field is implicitly fixed through the constraint equations (7.12) and (7.13). Our result shows that radiation is heading toward the black hole if the boundary conditions $\sigma_0 > 0$, $\dot{\sigma}_0 = 0$ are chosen.

Let us now turn to the second type of initial metric perturbation: $\sigma_0 = 0$, $\dot{\sigma}_0 > 0$. Here the spatial geometry is unperturbed on the initial slice, but it is given a kind of "push" that corresponds to a perturbation in the radiation bath. Solving once again for σ with these boundary conditions, and using Eq. (8.13), one finds for small t

$$\delta(t) = \dot{\sigma}_0 t^2 \tag{8.18}$$

The horizon perturbation grows. This perturbation mode is unstable, and leads to evaporation.

We have seen that the radiation equilibrium of a Nariai universe displays unusual and nontrivial stability properties. The evolution of the black hole horizon depends crucially on the type of metric perturbation. Nevertheless, one may ask whether a cosmological black hole will typically evaporate or not. Cosmological black holes cannot come into existence through classical gravitational collapse, since they live in an exponentially expanding de Sitter background. The only natural way for them to appear is through the quantum process of pair creation [2]. This pair creation process can also occur in an inflationary universe because of its similarity to de Sitter space [4, 3, 19]. The nucleation of a Lorentzian black hole spacetime is described as the analytic continuation of an appropriate complex solution of the Einstein equations, which satisfies the no-boundary condition [1]. We will show below that the no-boundary condition selects a particular linear combination of the two types of initial metric perturbation, thus allowing us to determine the fate of the black hole.

9. NO-BOUNDARY CONDITION

To obtain the unperturbed Euclidean Nariai solution in conformal gauge, one performs the analytic continuation $t = i\tau$ in the Lorentzian solution, Eq. (8.1). This yields

$$(ds^{\mathrm{IV}})^2 = e^{2\rho}(d\tau^2 + dx^2) + e^{-2\phi}\, d\Omega^2 \tag{9.1}$$

and

$$e^{2\rho} = \frac{1}{\Lambda_1}\frac{1}{\cosh^2\tau}, \qquad e^{2\phi} = \Lambda_2 \tag{9.2}$$

In four dimensions, this describes the product of two round two-spheres of slightly different radii, $\Lambda_1^{-1/2}$ and $\Lambda_2^{-1/2}$. The analytic continuation to a Lorentzian Nariai solution corresponds to a path in the τ plane, first along the real τ axis, from $\tau = -\infty$ to $\tau = 0$, and then along the imaginary axis from $t = 0$ to $t = \pi/2$. This can be visualized geometrically by cutting the first two-sphere in half and joining to it a Lorentzian (1 + 1)-dimensional de Sitter hyperboloid. Because the (τ, x) sphere has its north (south) pole at $\tau = \infty$ $(\tau = -\infty)$, it is convenient to rescale time:

$$\sin u = \frac{1}{\cosh \tau} \tag{9.3}$$

or, equivalently,

$$\cos u = -\tanh \tau, \qquad \cot u = -\sinh \tau, \qquad du = \frac{d\tau}{\cosh \tau} \tag{9.4}$$

With the new time coordinate u, the solution takes the form

$$(ds^{\mathrm{IV}})^2 = \frac{1}{\Lambda_1}(du^2 + \sin^2 u\, dx^2) + \frac{1}{\Lambda_2}\, d\Omega^2 \tag{9.5}$$

Now the south pole lies at $u = 0$, and the nucleation path runs to $u = \pi/2$ and then parallel to the imaginary axis $(u = \pi/2 + iv)$ from $v = 0$ to $v = \infty$.

The perturbation of $e^{2\phi}$, Eq. (8.6), introduces the variable σ, which satisfies the Euclidean version of Eq. (8.7):

$$\sin^2 u \frac{d^2\sigma}{du^2} + \sin u \cos u \frac{d\sigma}{du} - (1 - a \sin^2 u)\, \sigma = 0 \tag{9.6}$$

In addition, the nature of the Euclidean geometry enforces the boundary condition that the perturbation vanish at the south pole:

$$\sigma(u = 0) = 0 \tag{9.7}$$

Otherwise, $e^{2\phi}$ would not be single-valued, because the coordinate x degenerates at this point. This leaves $\dot{\sigma}$ as the only degree of freedom in the boundary conditions at $u = 0$.

It will be useful to define the parameter c by the relation $c(c + 1) \equiv a$. The classical case, $a = 2$, corresponds to $c = 1$; for small b, they receive the quantum corrections $a = 2 + b$ and $c = 1 + b/3$. With the boundary condition (9.7), the equation of motion for σ, Eq. (9.6), can be solved exactly only for integer c ($a = 2, 6, 12, 20, \ldots$). The solution is of the form

$$\sigma(u) = \sum_{0 \leq k < c/2} A_k \sin(c - 2k)u \tag{9.8}$$

with constants A_k. Even for noninteger c, however, this turns out to be a good approximation in the region $0 \leq u < \pi/2$ of the (u, v) plane. Since we are interested in the case where $b \ll 1$, the sum in Eq. (9.8) contains only one term, and we use the approximation[6]

$$\sigma(u) \approx \tilde{A} \sin cu \tag{9.9}$$

It is instructive to consider the classical case first. (Physically, this is questionable, since the no-boundary condition violates the constraints at second order in ϵ.) For $b = 0$, the solution $\sigma(u) = \tilde{A} \sin u$ is exact. Along the Lorentzian line ($u = \pi/2 + iv$), this solution becomes $\sigma(v) = \tilde{A} \cosh v$.

[6] Treating Eq. (9.6) perturbatively in b around $a = 2$ leads to untractable integrals. Fortunately the guessed approximation in Eq. (9.9) turns out to be rather accurate, especially for late Lorentzian times v, which is the regime in which we claim our results to be valid. It is easy to check numerically that for sufficiently large v ($v > 10$), both the real and the imaginary parts of Eq. (9.9) have a relative error $b/30$ or less. The result for the phase of the prefactor, Eq. (9.13), has a relative error of less than 10^{-4}, independently of b. Crucially, the exponential behavior at late Lorentzian times is reproduced perfectly, as the ratio

$$\frac{\partial\sigma/\partial v}{\sigma}$$

using the approximation, agrees with the numerical result to machine accuracy. Therefore the relative error in Eq. (9.15) is the same as in Eq. (9.9); in both equations it is located practically entirely in the magnitude of the prefactor. These statements hold for $0 \leq b \leq 1$, which really is a wider interval than necessary.

By transforming back to the Lorentzian time variable t, one can check that this is the stable solution found in the previous section, with $\sigma_0 = \tilde{A}$, $\dot{\sigma}_0 = 0$. For real $\tilde{A}$, it is real everywhere along the nucleation path. Thus, when the quantum fields are turned off, the Euclidean formalism predicts that the unstable mode will not be excited. This is a welcome result, since there are no fields that could transport energy from one horizon to another.

Once b is nonzero, however, it is easy to see that $\partial\sigma/\partial u$ no longer vanishes at the origin of Lorentzian time, $u = \pi/2$. This indicates that the unstable mode, $\dot{\sigma}_0 \neq 0$, will be excited. Unfortunately, checking this is not entirely straightforward, because σ is no longer real everywhere along the nucleation path. One must impose the condition that σ and $\dot{\sigma}$ be real at late Lorentzian times. We will first show that this can be achieved by a suitable complex choice of A. One can then calculate the horizon perturbation δ from the real late-time evolution of the metric perturbation σ to demonstrate that evaporation takes place.

From Eq. (9.9) one obtains the Lorentzian evolution of σ,

$$\sigma(v) = \tilde{A} \sin c \left(\frac{\pi}{2} + iv \right) \tag{9.10}$$

$$= \tilde{A} \left(\sin \frac{c\pi}{2} \cosh cv + i \cos \frac{c\pi}{2} \sinh cv \right) \tag{9.11}$$

For late Lorentzian times (i.e., large v), $\cosh cv \approx \sinh cv \approx e^{cv}/2$, so the equation becomes

$$\sigma(v) \approx \frac{1}{2} \tilde{A} \, (ie^{-ic\pi/2}) \, e^{cv} \tag{9.12}$$

This can be rendered purely real by choosing the complex constant $\tilde{A}$ to be

$$\tilde{A} = A \, (-ie^{ic\pi/2}) \tag{9.13}$$

where A is real.

Now we can return to the question of whether the Euclidean boundary condition leads to evaporation. After transforming the time coordinate we find that the expression for the growth of the horizon perturbation, Eq. (8.13), becomes

$$\delta(v) = \sigma \left[1 + \cosh^2 v \left(\frac{\partial\sigma/\partial v}{\sigma} \right)^2 \right]^{-1/2} \tag{9.14}$$

The late-time evolution is given by $\sigma(v) = \frac{1}{2} A\, e^{cv}$. This yields, for large v,

$$\delta(v) \approx \frac{A}{2}\, e^{cv} \left(1 + \frac{c^2}{4}\, e^{2v} \right)^{-1/2} \approx \frac{A}{c} \exp\left(\frac{b}{3}\, v \right) \tag{9.15}$$

This result confirms that pair-created cosmological black holes will indeed evaporate. The magnitude of the horizon perturbation is proportional to the initial perturbation strength A. The evaporation rate grows with $\kappa\Lambda$. This agrees with intuitive expectations, since κ measures the number of quantum fields that carry the radiation.

10. SUMMARY AND CONCLUSIONS

We have argued that the momentum representation of the wave function of the universe has several advantages over the metric representation. Most importantly, the requirement that we live in a Lorentzian universe can be implemented straightforwardly in this formulation: one must take the argument of the wave function to be purely imaginary. Moreover, unlike the three-metric, the canonical momentum is closely related to observable quantities like the expansion rate of the universe, and it distinguishes between expanding and contracting branches. While the momentum and metric representations are related by a Laplace transform and thus contain the same information, we conclude that many of the most relevant physical properties of a spacetime are manifest only in the momentum representation.

We have clarified how and under which conditions Euclidean solutions with a conical singularity may be used as saddlepoints. We showed that this is possible in the case of submaximal Schwarzschild–de Sitter universes if the spacelike boundary Σ is chosen to contain the conical singularity and the metric is specified there. On the rest of Σ, a purely imaginary second fundamental form is specified to ensure that the observed universe is Lorentzian. This enabled us to describe the quantum nucleation of such spacetimes and calculate their creation rate on a de Sitter background.

We have investigated the quantum stability of the Schwarzschild–de Sitter black holes of maximal mass, the Nariai solutions. From four-dimensional spherically symmetric general relativity with a cosmological constant and N minimally coupled scalar fields we obtained a two-dimensional model in which the scalars couple to the dilaton. The one-loop terms were included in the large-N limit, and the action was made local by introducing a field Z which mimics the trace anomaly.

We found the quantum-corrected Nariai solution and analyzed its behavior under perturbations away from degeneracy. There are two possible ways of specifying the initial conditions for a perturbation on a Lorentzian spacelike section. The first possibility is that the displacement away from the Nariai solution is nonzero, but its time derivative vanishes. This perturbation corresponds to nearly degenerate Schwarzschild–de Sitter space, and, somewhat surprisingly, this perturbation is stable at least initially. The second possibility is a vanishing displacement and nonvanishing derivative. These initial conditions lead directly to evaporation. The different behavior of these two types of perturbations can be explained by the fact that the initial radiation distribution is implicitly specified by the initial conditions, through the constraint equations.

If neutral black holes nucleate spontaneously in pairs on a de Sitter background, the initial data will be constrained by the no-boundary condition: it selects a linear combination of the two types of perturbations. By finding appropriate complex compact instanton solutions we showed that this condition leads to black hole evaporation. Thus neutral primordial black holes are unstable.

In a separate paper [20] it will be argued that one-loop quantum perturbations on the Nariai universe can lead to the formation of multiple black hole horizons. The evaporation of such black holes causes the spacelike sections of de Sitter space to dissociate into large, separate de Sitter universes. This effect occurs iteratively and leads to the proliferation of de Sitter universes.

REFERENCES

[1] J. B. Hartle and S. W. Hawking, *Phys. Rev. D* **28**, 2960 (1983).
[2] P. Ginsparg and M. J. Perry, *Nucl. Phys. B* **222**, 245 (1983).
[3] R. Bousso and S. W. Hawking, *Phys. Rev. D* **52**, 5659 (1995).
[4] R. Bousso and S. W. Hawking, *Phys. Rev. D* **54**, 6312 (1996).
[5] D. Garfinkle, S. B. Giddings, and A. Strominger, *Phys. Rev. D* **49**, 958 (1994).
[6] F. Dowker, J. P. Gauntlett, S. B. Giddings, and G. T. Horowitz, *Phys. Rev. D* **50**, 2662 (1994).
[7] S. W. Hawking, G. T. Horowitz, and S. F. Ross, *Phys. Rev. D* **51**, 4302 (1995).
[8] S. W. Hawking and S. F. Ross, *Phys. Rev. D* **52**, 5865 (1995).
[9] R. B. Mann and S. F. Ross, *Phys. Rev. D* **52**, 2254 (1995).
[10] W.-Z. Chao, *Int. J. Mod. Phys. D* **6**, 199 (1997).
[11] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1974).
[12] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, *Phys. Rev. D* **45**, 1005 (1992).
[13] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Lett. B* **292**, 13 (1992).
[14] J. G. Russo, L. Susskind, and L. Thorlacius, *Phys. Rev. D* **46**, 3444 (1992).
[15] R. Bousso and S. W. Hawking, *Phys. Rev. D* **56**, 7788 (1997).
[16] H. Nariai, *Sci. Rep. Tohoku Univ. Ser. I* **35**, 62 (1951).
[17] J. D. Hayward, *Phys. Rev. D* **52**, 2239 (1995).
[18] S. Nojiri and S. D. Odintsov, *Mod. Phys. Lett. A* **12**, 2083 (1997).
[19] R. Bousso, *Phys. Rev. D* **55**, 3614 (1997).
[20] R. Bousso, *Proliferation of de Sitter space*, submitted to *Phys. Rev. D* [hep-th/9805081].